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Kibble-Zurek problem: Universality and the scaling limit

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Near a critical point, the equilibrium relaxation time of a system diverges and any change of control/thermodynamic parameters leads to nonequilibrium behavior. The Kibble-Zurek problem is to determine the dynamical evolution of the system parametrically close to its critical point when the change is parametrically slow. The nonequilibrium behavior in this limit is controlled entirely by the critical point *and* the details of the trajectory of the system in parameter space (the protocol) close to the critical point. Together, they define a universality class consisting of critical exponents, discussed in the seminal work by Kibble and Zurek, and scaling functions for physical quantities, which have not been discussed hitherto. In this article, we give an extended and pedagogical discussion of the universal content in the Kibble-Zurek problem. We formally define a scaling limit for physical quantities near classical and quantum transitions for different sets of protocols. We report computations of a few scaling functions in model Gaussian and large-N problems and prove their universality with respect to protocol choice. We also introduce a protocol in which the critical point is approached asymptotically at late times with the system marginally out of equilibrium, wherein logarithmic violations to scaling and anomalous dimensions occur even in the simple Gaussian problem.

I. INTRODUCTION

The study of critical points and their associated continuum limits or field theories has been a central and enormously productive exercise in statistical mechanics and condensed matter physics. This study began with the problem of equilibrium finite-temperature transitions, was then extended to their dynamics, and thereafter to the interplay between criticality and finite-size effects.[1] This development has since been replayed in the theory of zero-temperature quantum phase transitions.[2,3] Needless to say, the development of powerful field-theoretic methods, most notably conformal field theory in $(1 + 1)$ dimensions[4] and most recently the gauge gravity duality,[5] have significantly advanced this line of work. In all these cases, the field-theoretic approach not only describes the critical point, but also the regions of the phase diagram adjacent to it.

A new dimension to the study of the passage through critical points was introduced by Kibble[6] in the context of the expanding universe, since recast in the language of critical phenomena by Zurek.[7] Their proposal, the "Kibble-Zurek mechanism," is a theory of the defects generated in a system being cooled through a continuous symmetry-breaking phase transition at a small, but finite rate. The system inevitably goes out of equilibrium on the approach to the transition and arrives in the broken-symmetry phase with different spatial regions realizing different orientations of the broken symmetry, and topological defects as a result. The mechanism predicts the scaling of the number of these defects with quench time[7,8] and has been tested in a variety of systems,[9–13] although quantitative agreement has been established only in zero-dimensional annular Josephson junctions[14] and in

pattern-forming steady-state transitions,[15,16] so there remains scope for more stringent experimental tests. It has been recently generalized to the setting of quantum phase transitions by Dziarmaga,[17,18] Polkovnikov,[19] and Zurek *et al.*,[20] where the role of temperature is played by a non-symmetry-breaking control parameter, and to ramps across multicritical points.[21,22] The scaling of other physical quantities such as excess heat with quench time has also been investigated.[23] Polkovnikov and co-workers have further paid attention to the interplay between finite system size and parameter velocity.

The general problem posed by the work of Kibble and Zurek is that of a slow passage through a critical or multicritical point, which we shall term the Kibble-Zurek (KZ) problem (often referred to by the oxymoronic term "slow quench"). This problem is then characterized by a critical point with its equilibrium physics and a "protocol," which is a particular path in the parameter space of the problem that touches the critical point. In the limit of asymptotically slow motion in parameter space, we expect that the physics is dominated by the critical point and hence is, in an appropriate sense, universal. An important task of theory is then to isolate this universal content and compute it. We should note that the study of universality in the KZ problem, especially in the quantum setting, is a part of a wider current study of nonequilibrium quantum dynamics; for broader perspective, see the recent discussion of Polkovnikov *et al..*[24]

In previous studies of critical phenomena, two ideas have proven extremely useful. The first is the idea of the scaling limit in which various quantities of interest, such as thermodynamic densities and correlation functions, are postulated to obey certain homogeneity relations. This set of scaling functions

along with the critical exponents capture the full universal content associated with a given critical point. The second is the renormalization group, which provides an understanding of the origin of this universality and a full computation of its content.

In this paper, we will make progress on the first front: we will formulate a scaling limit for the KZ problem and report model computations of the resulting scaling functions for a few classical and quantum problems. In appropriate limits, these scaling functions will reduce to those in the equilibrium and the coarsening problems. Except for recent work by Deng *et al.*,[25] Biroli *et al.*,[26] and De-Grandi *et al.*[27] discussing scaling functions in specific cases, the need for defining full scaling functions has been largely overlooked in the literature. Our contribution is to formalize the idea as a scaling limit for *all* physical quantities for any pairing of a critical point and a protocol.

The second part of the program, beyond our ambition at present, would be the construction of a renormalization group flow. Such a program has been fruitfully pursued in sudden quench studies in classical models with stochastic dynamics (see Ref. 28 and references therein). We offer two modest steps in that direction here. First, in the classical context, we formulate the path integral previously written only for sudden quenches to the KZ problem. This is, in principle, amenable to analysis by standard equilibrium renormalization group techniques. Second, we will prove universality with respect to protocol choice for some model classical and quantum problems. Specifically, we will show that the expectation that only the behavior of the protocols in the vicinity of the critical points is important is, in fact, correct.

In the course of this paper, we will offer a tripartite classification of possible protocols based on their "topology," i.e., whether they cross, turn around at, or end at the critical point. The scaling of quantities such as the defect density and the excess heat with quench time has been previously generalized[23,29] to arbitrary *positive* power-law behavior of the protocol on time near the critical point, that is, the first two kinds of protocols in our classification. Our contribution is the definition of a third class of power-law protocols that asymptotically end at the critical point and the identification of an interesting member, the marginal end critical protocol, that generates a one-parameter family of nonequilibrium deformations of the equilibrium critical state worthy of further study.

We turn now to the organization of this paper. We begin, in Sec. II, by defining the protocols of interest and introducing the well-known KZ time and length. In Sec. III, we lay out the scaling formalism for the KZ problem: the definition of the KZ scaling limit and the resulting scaling functions. Here, we also discuss how the KZ scaling functions universally interpolate between the early equilibrium physics and the late-time thermalization/coarsening physics. In Sec. IV, we use the $O(N)$ vector model endowed with Model A dynamics as our classical model system and compute scaling functions for it in the Gaussian and large-N approximations. We also give an explicit proof of protocol universality for this problem. In Sec. V, we turn to the quantum $O(N)$ model in the Gaussian approximation and compute various quantities of interest there. We conclude with some remarks in Sec. VI.

We note that a recent paper by Kolodrubetz *et al.* reports analogous results for the transverse-field Ising model in $(1 + 1)$ dimensions.[30] This work complements our own results on the quantum Ising universality class above the upper critical dimension $(3 + 1)$ where the critical theory is Gaussian.[31] We comment on the one case untreated in Ref. 30 later in the paper. The present authors will also report computations of scaling forms in an interacting quantum field theory using the tools provided by the Anti-de-Sitter/Conformal field theory (AdS/CFT) correspondence elsewhere.[32]

II. CLASSIFICATION OF PROTOCOLS AND THE KZ TIME AND LENGTH SCALES

Consider a multicritical point in d spatial dimensions. Let $\{O_i\}$ denote the set of relevant operators that couple to conjugate fields $\{h_i\}$ and have scaling dimensions $\{\Delta_i\}$. At the critical point, we set $\langle O_i \rangle_{eq} = 0$. The set of scaling dimensions $\{\Delta_i\}$ and a host of scaling functions constitute the static universal equilibrium content of this critical point. Along with the dynamical exponent z obtained from the time dynamics, they determine the universality class of the critical point. Let δ parametrize a path in the space of conjugate fields $\{h_i\}$ such that δ is zero at the critical point. For every such path, define the correlation length exponent ν to be $1/(d - \Delta)$, where Δ is the scaling dimension of the most relevant operator that has a projection along the path near $\delta = 0$. Along the path, the correlation length then diverges as $\delta^{-\nu}$ close to the critical point.

A useful example to keep in mind, particularly for the next section, is the ferromagnetic critical point. The relevant operators are the scalar energy operator and the vector magnetization, coupling to the conjugate fields of temperature and magnetic field, respectively. Unless mentioned otherwise, all length/time scales are dimensionless and are measured in units of some microscopic length/time. We also set $\hbar$ and k_B to 1.

A. Protocols in δ

Consider a system prepared in equilibrium at $t = -\infty$ at some fixed distance $\delta_- > 0$ away from the critical coupling $\delta = 0$ being evolved in time along the path $\delta(t)$ in conjugate field space. For simplicity, we restrict ourselves to paths with a unique tangent most relevant operator near $\delta = 0$. The KZ dynamics refines the classification of critical points discussed above using two pieces of data. The first is the symmetry of the path. If the most relevant operator along the path respects all the symmetries of the critical theory, the path is non-symmetry breaking, or else it is symmetry breaking. The second is the leading-order behavior of $\delta(t)$ near the critical point (strictly speaking, the critical coupling) which we classify in the following (see Fig. 1):

(i) Trans-critical protocols (TCPs): These protocols take the system across the critical point. They smoothly interpolate δ between $\delta_- > 0$ as $t \to -\infty$ and $\delta_+ < 0$ as $t \to \infty$, crossing the critical point at $t = 0$. An example of a TCP is

$$\delta(t; \tau_Q) = -\delta_0 \tanh \frac{t}{\tau_Q} . \tag{1}$$
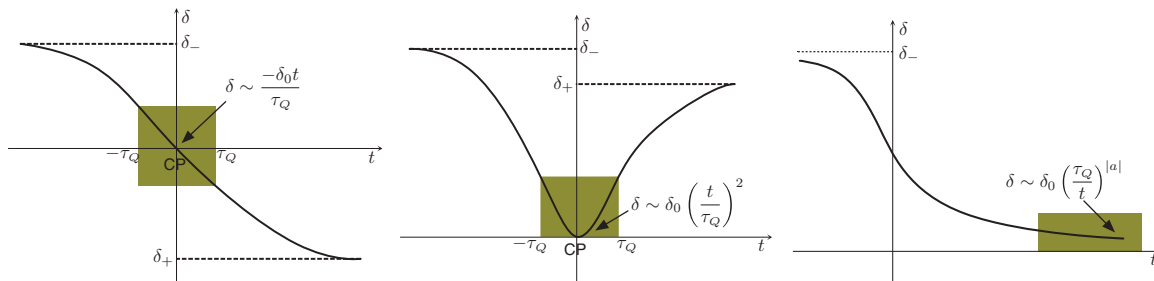
FIG. 1. (Color online) From left to right, examples of transcritical (TCP), cis-critical (CCP), and end-critical (ECP) protocols. The critical point is at $t = 0$. The leading-order expansion of δ is valid in the shaded region.

We will shortly show that the dynamic scaling functions are universal with respect to the behavior of the protocol near $t = 0$. In anticipation of this result, we classify the entire family of analytic TCPs by their leading-order behavior in a time scale τ_Q near $t = 0$ as

$$\delta(t; \tau_Q) = \begin{cases} \delta_-, & t \to -\infty \\ \delta_0 \left(\frac{-t}{\tau_q}\right)^a, & t \to 0 \\ \delta_+, & t \to \infty \end{cases} \quad (2)$$

with a odd. $a = 1$ is the linear protocol, $a = 3$ the cubic, and so on.

(ii) Cis-critical protocols (CCPs): These protocols keep the system in a single phase and touch the critical point at $t = 0$. They smoothly interpolate δ between δ_- as $t \to -\infty$ and $\delta_+ > 0$ as $t \to \infty$ through the critical point at $t = 0$. An example of a CCP is

$$\delta(t; \tau_Q) = \delta_0 \tanh^2 \frac{t}{\tau_Q} . \quad (3)$$

Equation (2) with a even classifies CCPs. $a = 2$ is the quadratic protocol, $a = 4$ the quartic, and so on.

(iii) End-critical protocols (ECPs): End-critical protocols (ECPs) keep the system in a single phase while asymptotically approaching the critical point. They smoothly interpolate between δ_- as $t \to -\infty$ and 0 as $t \to \infty$. The asymptotic approach to the critical point may be with or without a time scale. Here, we restrict ourselves to the family of scale-free protocols

$$\delta(t) = \begin{cases} \delta_-, & t \to -\infty \\ \delta_0 \left(\frac{t}{\tau_Q}\right)^a = \delta_0 \left(\frac{\tau_Q}{t}\right)^{|a|}, & t \gg \tau_Q \end{cases} \quad (4)$$

where $a < 0$ and τ_Q is the time scale over which the protocol behavior smoothly changes from being a constant to a power law. Unlike in the above two cases, a is not required to be integer valued.

B. KZ length and time

A system evolving from $t = -\infty$ by a TCP, CCP, or an ECP with large $|a|$ must fall out of equilibrium near the critical point due to critical slowing down. This is signaled by a diverging relaxation time ξ_t.[33] The time at which the system falls out of equilibrium is defined to be the KZ time t_Q. The KZ time defines a KZ length $l_Q \sim t_Q^{1/z}$. l_Q manifests in dynamic correlation functions as a crossover scale between equilibrium and nonequilibrium correlations.

More intuitively,[6,8] in an ordering transition, order is unable to form on scales larger than l_Q due to the finite quench rate, and domains of broken-symmetry phase of size l_Q persist in the ordered phase. We now separately consider the cases of TCP/CCP and ECP.

TCPs/CCPs. The scaling of the KZ length and time with τ_Q follows essentially from considerations of Refs. [20] and [29], which we recapitulate here. A purely quantum version can also be formulated using Landau-Zener arguments and perturbation theory.[19,20] We define $\xi(t; \tau_Q)$ and $\xi_t(t; \tau_Q)$ to be the instantaneous correlation length and time if the system were in equilibrium at $\delta(t; \tau_Q)$. The crucial quantities that determine t_Q are the change in the correlation time over a correlation time $\dot{\xi}_t \xi_t$ and ξ_t itself. When $\dot{\xi}_t \xi_t \ll \xi_t$ or $\dot{\xi}_t \ll 1$, ξ_t is changing slowly enough for the system evolution to be adiabatic. This is the case for $t < -\tau_Q$. As $\dot{\xi}_t$ diverges at the critical point, there must come a time $-t_Q > -\tau_Q$ when it is of order one:

$$\dot{\xi}_t(t_Q; \tau_Q) = 1 \Rightarrow t_Q \equiv \left(\frac{\tau_Q}{\delta_0^{1/a}}\right)^{\frac{a\nu z}{a\nu z + 1}}. \quad (5)$$

ECPs. The asymptotic behavior of $\dot{\xi}_t$ categorizes the ECPs into three. $|a_c| \equiv 1/(\nu z)$ as follows:

(a) Nonadiabatic: When $|a| > |a_c|$, $\dot{\xi}_t$ diverges at the critical point. The system falls out of equilibrium at t_Q given by Eq. (5) where a is now negative.

(b) Adiabatic: When $|a| < |a_c|$, $\dot{\xi}_t$ is zero at the critical point. When τ_Q is large, the evolution is adiabatic for all times.

(c) Marginal: When $|a| = |a_c|$, $\dot{\xi}_t$ is independent of t at the critical point and the only length scale in the problem is $t^{1/z}$. The system is marginally out of equilibrium on this scale in a sense that will become clear soon.

As we are interested in universal behavior *out of equilibrium*, $|a| \gtrsim |a_c|$ henceforth.

III. KZ SCALING LIMIT

The introduction of t_Q and l_Q is reminiscent of the introduction of finite-size cutoffs in the theory of equilibrium critical behavior, and we are led to analogs of the finite-size scaling limit and finite-size scaling functions. We define the KZ scaling limit to be the limit $\tau_Q \to \infty$ when time and length scales are measured in units of the diverging KZ scales, t_Q and l_Q. $\delta(t; \tau_Q) \to 0$ in this limit, and the system is arbitrarily close to the critical point, evolving nonadiabatically for all t/t_Q
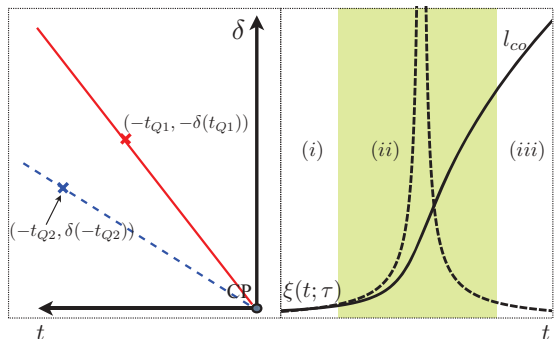
FIG. 2. (Color online) Left: δ as a function of t close to $t = 0$ with KZ times shown for a slow (2, dashed line) and fast (1, solid line) quench. In the slower quench, the system falls out of equilibrium earlier ($|t_{Q2}| > |t_{Q1}|$), but at a smaller distance from the critical point [$\delta(-t_{Q2}) < \delta(-t_{Q1})$]. Right: The correlation length of a system quenched at a finite rate (solid line) vs $\xi(t; \tau_\varrho)$ (dashed line). KZ split dynamics into (i) adiabatic, (ii) sudden, and (iii) post-quench (here coarsening) regimes. The KZ scaling limit describes (ii) with (i) and (iii) as asymptotes.

[the shaded region in Fig. 2(b)]. The marginal ECP is special and we discuss it separately at the end of this section.

We now turn to the definition of the scaling functions that arise in the KZ scaling limit. We discuss the scaling functions of various physical quantities and their asymptotic forms in specific cases. Any readers new to scaling theory are encouraged to arm themselves with the Model A example in the Gaussian limit discussed in Sec. IV and interpret the definitions below in that context.[1] We initially consider classical critical points and discuss asymptotic forms when the transition is ordering. We then comment on the special features of quantum critical points.

Our notation is to denote the absolute value of any vector **k** by k, and to use $\bar{x}$ and $\bar{t}$, respectively, to refer to the scaled length and time x/l_ϱ and t/t_ϱ. We reserve calligraphic lettering for scaling functions.

A. TCPs and CCPs

1. Scaling forms of correlation functions

Consider the scalar operator O with scaling dimension Δ. We assume that the theory has translational and rotational invariance.[34] The KZ scaling forms for the one- and two-point connected correlation functions are

$$\langle O(\mathbf{x},t) \rangle_{\tau_\varrho} \equiv G_O(t; \tau_\varrho) \sim \frac{1}{l_\varrho^\Delta} \mathcal{G}_O \left(\frac{t}{t_\varrho} \right),$$

$$\langle O(\mathbf{x},t) O(\mathbf{x}',t') \rangle_{\tau_\varrho} \equiv G_{OO}(|\mathbf{x} - \mathbf{x}'|, t, t'; \tau_\varrho)$$

$$\sim \frac{1}{l_\varrho^{2\Delta}} \mathcal{G}_{OO} \left(\frac{|\mathbf{x} - \mathbf{x}'|}{l_\varrho}, \frac{t}{t_\varrho}, \frac{t'}{t_\varrho} \right). \quad (6)$$

When we use "$\sim$" to indicate a scaling form of a correlation, as in Eq. (6), we have a precise limiting statement in mind. For example, in the case of the two-point connected correlator, what we mean is

$$\lim_{\tau_\varrho \to^* \infty} l_\varrho^{2\Delta} G_{OO}(x, t, t'; \tau_\varrho) = \mathcal{G}_{OO} \left(\frac{x}{l_\varrho}, \frac{t}{t_\varrho}, \frac{t'}{t_\varrho} \right), \quad (7)$$

where $\tau_\varrho \to^* \infty$ means the limit where $\tau_\varrho \to \infty$ with $\frac{x}{l_\varrho}$, $\frac{t}{t_\varrho}$, and $\frac{t'}{t_\varrho}$ held fixed.

The scaling forms of all higher-order cumulants and cross correlators with other relevant operators are straightforward extensions of the form in Eq. (6). Note that $\mathcal{G}_O$ can be identically zero. This is the case, for instance, in a zero-field temperature quench through the ferromagnetic critical point when O is the spin operator. The finite ramp rate prevents order from forming on scales longer than the KZ length, and $\langle O \rangle_{\tau_\varrho}$, the average magnetization, remains zero at all times.

2. Asymptotic form near equilibrium

By construction of the protocol, we should recover equilibrium scaling forms in certain limits as $\bar{t} \to \pm\infty$. However, the precise limits are subtle and we derive them in the following. Recall that the equilibrium scaling limit is the limit of $\delta \to 0$ holding x/ξ and $(t - t')/\xi^z$ fixed, wherein the two-point correlator has the scaling form

$$\langle O(\mathbf{x},t) O(\mathbf{0},t') \rangle_\delta^{\text{eq}} \sim \xi^{-2\Delta} \mathcal{G}_{OO}^{\text{eq}} \left(\frac{x}{\xi}, \frac{t - t'}{\xi^z} \right). \quad (8)$$

The "$\sim$" symbol here is distinct from that in the previous section.

We consider the limit in which the KZ scaling form [Eq. (7)] reduces to the equilibrium scaling form [Eq. (8)]. First, we observe that the relation

$$\frac{x/l_\varrho}{x/\xi(t; \tau_\varrho)} = \frac{\xi(t; \tau_\varrho)}{l_\varrho} = \left| \frac{t}{t_\varrho} \right|^{-av} \quad (9)$$

implies that x/ξ is fixed whenever $\bar{x}$ and $\bar{t}$ are held fixed. Similarly, t/ξ^z and t'/ξ^z are also fixed by $\bar{t}$ and $\bar{t}'$. Thus, an alternative to the KZ scaling form (7) involving the same two arguments as the equilibrium scaling form may be given as

$$\lim_{\tau_\varrho \to^* \infty} \xi^{2\Delta} G_{OO}(x, t, t'; \tau_\varrho) = \mathcal{G}_{OO}^{(2)} \left(\frac{x}{\xi}, \frac{t - t'}{\xi^z}, \frac{t + t'}{t_\varrho} \right). \quad (10)$$

When $|t| \gg t_\varrho$, the system is in instantaneous equilibrium on length and time scales ξ and ξ_t, respectively. This is the limit in which $\mathcal{G}_{OO}^{(2)}$ must reduce to $\mathcal{G}_{OO}^{\text{eq}}$. Thus, in the limit $(\bar{t} + \bar{t}') \to \pm\infty$ holding x/ξ and $(t - t')/\xi^z$ fixed,

$$\mathcal{G}_{OO}^{(2)} \left(\frac{x}{\xi}, \frac{t - t'}{\xi^z}, \frac{t + t'}{t_\varrho} \right) \sim \mathcal{G}_{OO}^{\text{eq}} \left(\frac{x}{\xi}, \frac{t - t'}{\xi^z} \right). \quad (11)$$

For the original KZ scaling form (7), this translates to the requirement

$$\mathcal{G}_{OO}(\bar{x}, \bar{t}, \bar{t}') \sim \bar{t}^{2av\Delta} \mathcal{G}_{OO}^{\text{eq}}[\bar{x} \, \bar{t}^{va}, (\bar{t} - \bar{t}') \bar{t}^{zav}] \quad (12)$$

in the same limit. As expected, $\xi(t; \tau_\varrho)/\xi(t'; \tau_\varrho) \to 1$ so that there is a single diverging length in the equilibrium system. Furthermore, time-translation invariance is recovered.

In the example of a temperature quench through a ferromagnetic critical point, the equilibrium equal-time connected spin-spin correlation function decays exponentially on a length scale ξ on either side of the critical point, consequently, Eq. (12), when $\bar{t} = \bar{t}'$ must asymptote to

$$\mathcal{G}_{OO}(\bar{x}, \bar{t}, \bar{t}) \sim \bar{t}^{2av\Delta} \exp(-\bar{x} \bar{t}^{va}).$$

3. Asymptotic form for coarsening dynamics

It is generally believed[35] that a system quenched to an ordered phase with multiple vacua undergoes coarsening, whereby each local broken-symmetry region grows in time and the system is asymptotically statistically self-similar on a characteristic length scale $l_{co}(t) \gg \xi$. Put another way, the two-point function heals to its equilibrium value on the scale ξ within each "domain," and is exponentially suppressed between domains, each of growing length $l_{co} \gg \xi$. In the late-time regime, dynamical scaling is expected to hold when there are no growing scales competing with l_{co}. For example, as $t, t' \to \infty$,

$$\langle O(\mathbf{x}, t) O(\mathbf{0}, t') \rangle_\delta \sim \xi^{-2\Delta} \mathcal{G}^{co}_{OO}\left(\frac{x}{l_{co}(t)}, \frac{x}{l_{co}(t')}\right),$$

$$\text{where} \quad l_{co}(t) \equiv t^\theta = t^{-a\nu + \frac{a\nu z + 1}{z_d}}. \tag{13}$$

z_d is a dynamic exponent specific to coarsening. The coarsening scaling forms in various models are reviewed in Ref. 36.

When present in the KZ problem, we expect coarsening physics to emerge deep in the ordered phase, i.e., as $\bar{t}, \bar{t}' \to \infty$, on the length scales $l_{co}(t), l_{co}(t')$. Proceeding as in the previous section, we conclude that as $\bar{t}, \bar{t}' \to \infty$ holding $x/l_{co}(t)$ and $x/l_{co}(t')$ fixed, the two-point function must have the limiting form

$$\mathcal{G}_{OO}(\bar{x}, \bar{t}, \bar{t}') \sim \bar{t}^{2a\nu\Delta} \mathcal{G}^{co}_{OO}\left(\frac{\bar{x}}{\bar{t}^\theta}, \frac{\bar{x}}{\bar{t}'^\theta}\right).$$

Note that the limiting equilibrium form requires holding $x/\xi(t)$ fixed and that $l_{co}(t)/\xi(t)$ diverges as $\bar{t} \to \infty$.

In the simplest case of Model A dynamics for spins with N components, $z_d = z = 2$ and $l_{co}(t) \sim \sqrt{t}$ in the infinite-N limit. Although this phenomenology holds for systems with and without topological defects, the specific form of the equal-time spin-spin correlation function depends on the presence of topological defects.[36]

4. Scaling form of the nonequilibrium correlation length

The nonequilibrium correlation length ξ_{ne} is defined to be the inverse of the decay constant on the longest length scales of the two-point equal-time correlator in real space. In equilibrium, this length is the crossover scale in correlation functions between fluctuations dominated by one fixed point and another. A particularly simple definition of ξ_{ne} is

$$\xi_{ne}(t; \tau_\varrho) = \sqrt{\frac{\int dx^d \, x^2 G_{OO}(x, t, t; \tau_\varrho)}{\int dx^d \, G_{OO}(x, t, t; \tau_\varrho)}}, \tag{14}$$

and is useful when G_{OO} is always positive. A more general definition is through the smallest imaginary part of the poles of $G_{OO}(k, t, t; \tau_\varrho)$ in k space; we will elaborate on this when the necessity arises. Biroli and coauthors[26] recently discussed the KZ scaling form of ξ_{ne} in non-symmetry-breaking TCPs. They observed that ξ_{ne} must asymptote to the equilibrium correlation length $\xi(t; \tau_\varrho)$, as $\bar{t} \to -\infty$; that it must scale according to the critical coarsening form $t_\varrho^{1/z}$, when $|\bar{t}| \sim O(1)$; and that it must asymptote to the coarsening length l_{co} as $\bar{t} \to \infty$. Their proposed scaling form can be derived from that of G_{OO} and

can be rewritten as

$$\xi_{ne}(t; \tau_\varrho) \sim l_\varrho \mathcal{L}_{ne}\left(\frac{t}{t_\varrho}\right). \tag{15}$$

The asymptotic behavior of ξ_{ne} for $\hat{t} \to \pm\infty$ formally translates to the limits

$$\mathcal{L}_{ne}(x) \sim \begin{cases} |x|^{-a\nu}, & x \to -\infty \\ |x|^\theta, & x \to \infty. \end{cases} \tag{16}$$

Absent coarsening physics, the right asymptote reproduces the instantaneous correlation length.

5. Scaling form of the number of defects

An intuitively appealing picture of the lack of order on length scales greater than the KZ length is through topologically protected point defects of characteristic separation l_ϱ, and/or defects of dimension p with characteristic separation l_ϱ in any hyperplane of codimension p. This picture is really only meaningful if the separation between defects is much larger than the equilibrium correlation length, a constraint met only in the coarsening regime $t \gg t_\varrho$. In the coarsening regime, the density of a defect of dimension p in the hyperplane of codimension p should scale as $1/l_{co}(t; \tau_\varrho)^{d-p}$. At fixed positive $\bar{t}$, this reproduces the celebrated scaling of the defect density with τ_ϱ,

$$\text{Density of defects} \sim \left(\frac{1}{l_\varrho}\right)^{d-p} \sim \tau_\varrho^{\frac{-a\nu(d-p)}{a\nu z + 1}}. \tag{17}$$

The above scaling with τ_ϱ has been verified in experiments in zero-dimensional annular Josephson junctions[14] and in nonlinear optical and hydrodynamical systems undergoing steady-state transitions.[15,16] It is worth noting that the frequently cited experiments[37,38] in liquid-crystal systems do not test this critical scaling, but instead test the coarsening scaling forms. Although we have presented the above as a natural scaling ansatz, it may be derivable from the known scaling of correlation functions, for example, by the methods of Halperin-Liu-Mazenko for classical transitions.[39,40]

6. Scaling forms of thermodynamic analogs

The dynamics of classical systems is typically modeled phenomenologically by stochastic differential equations, possibly with conservation laws.[41] It is often useful to reformulate the stochastic dynamics in d dimensions in terms of a path integral in $(d + 1)$.[42,43] For Gaussian noise, the generating functional of correlation functions of the fundamental field $O(x, t)$ is

$$Z[J] = \int \mathcal{D}O(x, t) \exp\left(-\int d^d x \, dt \, [\mathcal{L} + J(x, t) O(x, t)]\right).$$

Here, J is the source for O, and $\mathcal{L}$ is a local Lagrangian density in which parameters of the protocol (like τ_ϱ) appear as couplings. When hyperscaling is obeyed, all KZ scaling forms follow from the scaling of the associated free energy.

In this formulation, it is natural to define a time-dependent free-energy density. We divide the $(d + 1)$-dimensional spacetime into a stack of spatial slices with volume L^d and temporal length Δt. Neglecting boundary effects, we compute the

free-energy density of a slice:

$$f(t;\tau_\varrho) = \lim_{\substack{L \to \infty \\ \Delta t \to 0}} \frac{-\ln(Z)}{L^d \Delta t}, \tag{18}$$

where

$$Z(t) = \int \mathcal{D}O \exp\left(-\int_{\mathbf{x} \in [-L,L]^d} d^d x \int_t^{t+\Delta t} dt \, \mathcal{L}\right).$$

For time independent δ, f is also time independent, and its leading nonanalytic dependence on δ is $f_{na} \sim \delta^{vd}$. When δ varies with time, we conjecture a scaling form for $f_{na}(t;\tau_\varrho)$ in the KZ limit:

$$f_{na}(t;\tau_\varrho) \sim \frac{1}{l_\varrho^d} \mathcal{F}\left(\frac{t}{t_\varrho}\right). \tag{19}$$

When $t/t_\varrho \ll -1$, the system is asymptotically in equilibrium. Thus,

$$\mathcal{F}\left(\frac{t}{t_\varrho}\right) \sim \left(\frac{t}{t_\varrho}\right)^{avd} \quad \text{when} \quad t/t_\varrho \ll -1. \tag{20}$$

This is also the expected asymptotic form of $\mathcal{F}$ if the system does not coarsen for $t/t_\varrho \gg 1$.

B. ECPs

The definition of the scaling limit, the KZ scaling forms of various observables, and their asymptotic behavior when the system is subjected to a nonadiabatic ECP closely follow the TCP discussion. Qualitatively, note the following:

(i) When $\bar{t} \to 0$, the system evolution is adiabatic and equilibrium scaling forms are recovered on the scale of the instantaneous correlation length and time.

(ii) The long-time behavior of a sudden quench to the critical point is recovered in the limit $\bar{t} \to \infty$. On a growing length scale $l_{co}(t) = t^{1/z}$, the system appears self-similar and satisfies dynamical scaling [Eq. (13)]. On length scales x much smaller than $l_{co}(t)$, correlations have relaxed to their critical form, while for $x \gg l_{co}(t)$, the system coarsens.

(iii) Scaled times near 1 probe a universal early-time regime of a sudden quench, i.e., $t \ll \xi_{t,0}$ where $\xi_{t,0}$ is the relaxation time before the quench. In the sudden quench, this regime is tied to boundary criticality.[44]

When $|a|vz = 1$ in Eq. (4), $\dot{\xi}_t$ is a constant for $t > \tau_\varrho$. By suitably redefining constants, Eq. (4) can be rewritten as

$$\delta(t) = \begin{cases} \delta_-, & t \to -\infty \\ \left(\frac{\theta}{t}\right)^{1/vz}, & t \gg \tau_\varrho. \end{cases} \tag{21}$$

The key distinction between the marginal ECP and all the other protocols is that there is only one growing length scale $t^{1/z}$ everywhere in the power-law regime. At late times, we expect the system to appear self-similar on this scale. This leads us to the scaling limit $x,t \to \infty$ at fixed θ holding $x/t^{1/z}$ fixed:

$$G_O(t) \sim \frac{1}{t^{\Delta/z}},$$

$$G_{OO}(x,t,t') \sim \frac{1}{t^{2\Delta/z}} \mathcal{G}_{OO}\left(\frac{x}{t^{1/z}}, \frac{t}{t'}\right). \tag{22}$$

We emphasize that the scaling functions above do not connect to the adiabatic limit. Unfortunately, this also implies that they necessarily contain some nonuniversal data. The source of this nonuniversality will be clearer in Sec. V F, where we are nevertheless able to identify interesting nonequilibrium behavior.

C. Quantum systems

We comment in the following on the major differences that arise in the treatment of quantum systems.

(i) The scaling of the correlation functions proceeds as before. However, the analogs of thermodynamic quantities of interest for protocols that begin with the system at $T = 0$ are now the excitation energy density in excess of the energy density in the adiabatic ground state, or "heat" density,[23,45] and the entropy density for which plausible definitions can be constructed from the diagonal entropy[46,47] or the entanglement entropy of a macroscopic subregion. These are, respectively, expected to exhibit the scaling forms

$$q(t;\tau_\varrho) \sim \frac{1}{l_\varrho^{d+z}} \mathcal{Q}\left(\frac{t}{t_\varrho}\right), \tag{23}$$

$$s(t;\tau_\varrho) \sim \frac{1}{l_\varrho^d} \mathcal{S}\left(\frac{t}{t_\varrho}\right). \tag{24}$$

For an isolated quantum system, by construction of the protocol, s and q are constant as $t \to -\infty$. As $\bar{t} \to \infty$, s and q also tend to a constant provided the system thermalizes to the Gibbs or the generalized Gibbs ensemble.

(ii) For protocols that begin with the system in equilibrium at $T > 0$, a new dimensionless parameter $k_B T / (\hbar/t_\varrho)$ now enters the quantum problem. Along with the quantities held fixed as the quench time is taken to be arbitrarily large in Sec. III A 1, we also hold $T t_\varrho$ fixed. For example, the scaling form for the two-point unequal time correlation function is now

$$\langle O(0,t)O(\mathbf{x}',t') \rangle_{\tau_\varrho,T} \sim \frac{1}{l_\varrho^{2\Delta}} \mathcal{G}_{OO}\left(\frac{x'}{l_\varrho}, \frac{t}{t_\varrho}, \frac{t'}{t_\varrho}, T t_\varrho\right).$$

The definition of the entropy goes through as before, but the excitation energy density is now measured with respect to the energy density that would be obtained in a strictly adiabatic evolution.[48]

(iii) Integrable quantum systems allow the existence of sharply defined quasiparticle excitations; their density resolved by momentum can serve as a uniquely quantum observable for CCPs and ECPs. When the vacuum is not unique, it is not straightforward to define quasiparticles and separate them from domain walls and other topological defects. Currently, we are not aware of an integrable system above one dimension where this question can be properly posed[49] (in one dimension, there is no real distinction between quasiparticles and domain walls). Note that the definition of the thermodynamic quantities of heat and entropy do not rely on integrability. For instance, in a cyclic process such as a CCP, $q(t;\tau_\varrho)$ when $t \gg t_\varrho$ is the difference of the system's energy density between symmetric time points.

D. Comments

We conclude this section with four comments.

(i) The Kibble-Zurek picture was initially proposed for a linear ramp through a thermal transition. The time dynamics was decoupled into three regions: the adiabatic regime for $t < -t_\varrho$, the diabatic or sudden or impulse region from $-t_\varrho$ to t_ϱ when the system is frozen, and the post-quench regime for $t > t_\varrho$ when the system is unfrozen and evolves through domain growth,[8] defect-antidefect annihilation, etc. Recently,[26] this picture was extended to account for evolution in the impulse regime using critical coarsening results. The KZ scaling forms introduced in this article also probe dynamics in the impulse regime with the adiabatic and the post-quench regimes acting as asymptotic limits.

(ii) A finite system dimension L can be readily accommodated in our scaling forms in the combination L/l_ϱ. The results obtained through adiabatic perturbation theory[19,50] can therefore be fitted within this framework.

(iii) We expect that the KZ scaling forms are unchanged by the inclusion of interaction terms in the dynamics that are irrelevant to the critical theory.[51] In the classical setting, the reformulation of the dynamical problem in d dimensions as an inhomogeneous statistical problem in $(d + 1)$ dimensions allows us to establish this by standard power-counting arguments. The quantum case is more subtle and needs to be treated case by case.

(iv) As the free Gaussian theory is the stable critical point in $d > 4$, the KZ scaling forms that we calculate in the following in that theory are universal for $d > 4$. In the classical setting, interaction terms in the Gaussian theory can be treated as perturbations to the action defined in Sec. III A 6 and will modify dynamics only on time scales t_{int} that are parametrically larger than t_ϱ. In the quantum case, these terms induce scattering between the free quasiparticles of the unperturbed theory on time scales t_{int} that are parametrically larger than t_ϱ. In either case, $t_{\text{int}}/t_\varrho \to \infty$ in the scaling limit, and the effects of these terms drop out.

IV. CLASSICAL SYSTEMS WITH MODEL A DYNAMICS

We illustrate the universality of the KZ scaling limit and explicitly compute scaling functions of a vector operator $\vec{\phi}$ in the simplest setting: Model A (Ref. 41) dynamics with a Landau-Ginzburg-Wilson (LGW) free energy. Model A dynamics is dissipative and obeys no conservation laws. Let $\vec{\phi}$ be an N component vector field in $d = 3$ dimensions. The dimensionless LGW free energy and the equation of motion are, respectively,

$$F = \int d^3x \left[\frac{1}{2}\left(|\boldsymbol{\nabla}\vec{\phi}|^2 + r_0|\vec{\phi}|^2 + \frac{u}{2N}|\vec{\phi}|^4 \right) - \sqrt{N}h^\alpha \phi_\alpha \right],$$
$$\frac{\partial \phi_\alpha}{\partial t} = -\frac{\partial F}{\partial \phi_\alpha} + \zeta_\alpha. \tag{25}$$

x and t are dimensionless and measured in units of the inverse cutoff Λ^{-1} and Λ^{-z}, respectively. ζ_α is a zero-mean spatially uncorrelated white-noise stochastic variable for every α. The variance of ζ_α is chosen to be 2 so that the long-time limit of the structure factor computed from the equation of motion when F is time independent is equal to the equilibrium structure factor:

$$\langle \zeta_\alpha(\mathbf{x},t)\zeta_\beta(\mathbf{x}',t') \rangle = 2\delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t'). \tag{26}$$

A. Gaussian limit

In this limit, we drop the ϕ^4 term in F. The critical point is at $r_0 = 0, h^\alpha = 0$, i.e., at the origin in the (r_0, h^α) parameter space. Let us restrict ourselves to paths in this space that lie along the axes and include the origin. The equilibrium theory is sensible only at the origin and when $r_0 > 0$. We can therefore study CCPs and ECPs along the non-negative r_0 axis ($\delta = r_0$). Luckily, we can also study TCPs along the same axis as the time-dependent fields are finite even when $r_0 < 0$. This physically uninteresting protocol is pedagogically useful. The critical exponents for all such paths are $\nu = 1/2, z = 2$. In the remainder of the discussion, $\delta = r_0$ and $h^\alpha, u = 0$.

Equation (25) is linear in $\vec{\phi}$ and hence diagonal in Fourier space. It can be explicitly solved for any protocol $\delta(t; \tau_\varrho) = r_0(t; \tau_\varrho)$ and a fixed noise realization:

$$\phi(\mathbf{k},t) = \int_{-\infty}^{t} dt'\, e^{-\int_{t'}^{t} dt''[k^2 + r_0(t''; \tau_\varrho)]} \zeta(\mathbf{k},t'). \tag{27}$$

We have dropped the component label of $\vec{\phi}$ for brevity. Memory of the initial condition at $t = -\infty$ is lost on the time scale $1/r_0(-\infty; \tau_\varrho)$ and is therefore absent in the solution above. The equal-time structure factor for each component, defined as $\langle \phi(\mathbf{k},t)\phi(\mathbf{k}',t) \rangle = (2\pi)^3\delta^3(\mathbf{k} + \mathbf{k}')G_{\phi\phi}(k,t; \tau_\varrho)$, is

$$G_{\phi\phi}(k,t; \tau_\varrho) = 2\int_{-\infty}^{t} dt'\, e^{-2\int_{t'}^{t} dt''[k^2 + r_0(t''; \tau_\varrho)]}. \tag{28}$$

1. Scaling limit for tanh TCP

The universality of the scaling limit with respect to details of the the protocol is already apparent when we consider the simple TCP $r_0(t; \tau_\varrho) = -\tanh(t/\tau_\varrho)$. This is an example of a protocol that is linear near the critical point, i.e., $a = 1$ in Eq. (2). Consequently, the KZ time and length scales are

$$t_\varrho = \sqrt{\tau_\varrho}, \quad l_\varrho = \tau_\varrho^{1/4}.$$

Rewriting Eq. (28) in units $\bar{k} = kl_\varrho$ and $\bar{t} = t/t_\varrho$,

$$G_{\phi\phi}(k,t; \tau_\varrho) = 2t_\varrho \int_{-\infty}^{\bar{t}} d\bar{t}' e^{-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}''[\bar{k}^2 - t_\varrho \tanh(\bar{t}''/t_\varrho)]}.$$

As $\tau_\varrho \to \infty$, $t_\varrho \tanh(\bar{t}''/t_\varrho) \to \bar{t}''$ and the KZ scaling form of the two-point equal-time correlation function is

$$G_{\phi\phi}(k,t; \tau_\varrho) \sim l_\varrho^2 \mathcal{G}_{\phi\phi}(\bar{k},\bar{t})$$
$$\sim l_\varrho^2 \int_{-\infty}^{\bar{t}} 2\, d\bar{t}' e^{-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}''(\bar{k}^2 - \bar{t}'')}. \tag{29}$$

Observe that the scaling function only depends on the leading behavior of the protocol near the critical point. For all protocols such that $r_0(t; \tau_\varrho) \sim -t/\tau_\varrho$ in the vicinity of zero time, the scaling form of the structure factor is the expression that we have just derived.

2. Scaling functions for all TCPs and CCPs

Wick's theorem informs us that all higher-order cumulants of ϕ only depend on $G_{\phi\phi}$. The details of the protocol enter the expression of $G_{\phi\phi}$ only in the combination $t_\varrho r_0(t; \tau_\varrho)$. In the
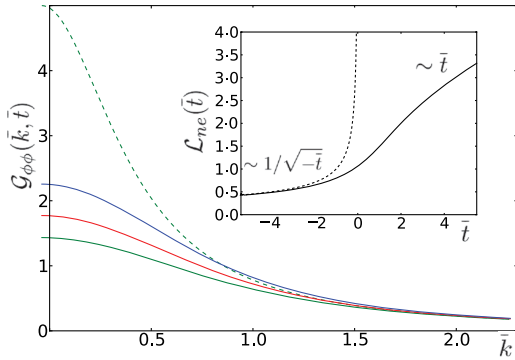
FIG. 3. (Color online) $\mathcal{G}_{\phi\phi}(\bar{k},\bar{t})$ vs $\bar{k}$ at fixed time slices for the linear TCP. The blue, red, and green solid lines are, respectively, at $\bar{t} = 0.2$, 0, and -0.2. The green dashed line is the correlator if the system were in equilibrium at $\bar{t} = -0.2$. Inset: The scaling form of the nonequilibrium correlation length vs $\bar{t}$ in solid. The dashed line is the instantaneous correlation length in units of l_{ϱ}, $\xi(t;\tau_{\varrho})/l_{\varrho}$.

scaling limit, for any a in Eq. (2), it is easily seen that

$$t_{\varrho} r_0(t;\tau_{\varrho}) \sim \left(\frac{-t}{t_{\varrho}} \right)^a . \tag{30}$$

The scaling function of the equal-time correlation function for any protocol with the near-zero behavior in Eq. (2) is therefore

$$\mathcal{G}_{\phi\phi}(\bar{k},\bar{t}) = 2 \int_{-\infty}^{\bar{t}} d\bar{t}' \, e^{-2 \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' [\bar{k}^2 + (-\bar{t}'')^a]} . \tag{31}$$

As seen from Fig. 3 for the linear TCP, $\mathcal{G}_{\phi\phi}$ and $\mathcal{L}_{\mathrm{ne}}$ match equilibrium forms as $\bar{t} \to -\infty$ and $\bar{k} \gg 1$:

$$l_{\varrho}^2 \mathcal{G}_{\phi\phi}(\bar{k},\bar{t}) \sim \xi^2 \, \mathcal{G}_{\phi\phi}^{\mathrm{eq}}[k\xi(t)] \equiv \frac{\xi(t)^2}{k^2 \xi(t)^2 + 1},$$

$$\xi_{\mathrm{ne}} \sim l_{\varrho} \mathcal{L}_{\mathrm{ne}} \equiv \frac{1}{(-t)^{1/2}} . \tag{31}$$

Observe that $\mathcal{G}_{\phi\phi}(0,0)$ is finite, indicating the suppression of order on length scales longer than l_{ϱ}. As $\bar{t} \to \infty$, the nonequilibrium correlation length grows without bound for TCPs due to the inverted potential. For CCPs, on the other hand, all scaling forms asymptote to the equilibrium forms at large positive times.

In the Appendix, we compute the partition function in $(3+1)$ dimensions and demonstrate the validity of the scaling hypothesis for the free-energy density f_{na}.

3. Scaling functions for the ECPs

The scaling functions of the nonadiabatic ECPs are identical to those of the TCPs/CCPs when $a < -1/\nu z$. Their asymptotic behavior may also be verified to be in agreement with Sec. III B. The more interesting case is the marginal ECP $r_0(t) = \theta/t$. Independent of the early-time regularization on the time scale τ_{ϱ}, $G_{\phi\phi}$ has the asymptotic form in Eq. (22):

$$G_{\phi\phi}(k,t) \sim t \, \mathcal{G}_{\phi\phi}(k\sqrt{t}),$$

$$\mathcal{G}_{\phi\phi}(x) \equiv \frac{4 e^{-2x^2}}{x^{4\theta+2}} \int_0^x dy \, e^{2y^2} y^{4\theta+1} .$$

As promised, $\mathcal{G}_{\phi\phi}(x)$ is distinct from $\mathcal{G}_{\phi\phi}^{\mathrm{eq}}(x)$ and the nonequilibrium correlation length is a multiple of the instantaneous

one $\xi(t) = \sqrt{t/\theta}$ at late times. The system exhibits critical coarsening and relaxes to the critical point as $t \to \infty$.

B. Large-N limit

In the infinite-N limit, the LGW theory is exactly solvable in terms of one- and two-point correlators. All higher-order correlators follow by the application of Wick's theorem. The one- and two-point correlators are known exactly in equilibrium and can be reduced to quadrature with Model A dynamics. The theory in $d = 3$ that we discuss here exhibits nontrivial critical behavior and affords us a probe of symmetry-breaking paths and coarsening physics.

Assume in Eq. (25) that $h^i \neq 0$ for $i = 1$ and 0 otherwise. This specifies all one-point correlators for $i > 1$ to be zero. The two-point correlator for every component is the same and is a function of the variable $m^2 = r_0 + u\langle \phi_{\alpha}^2 \rangle$. The well-known self-consistency equations relating $\langle \phi_1 \rangle$, $G_{\phi\phi}$, and m^2 are

$$r_0 + u \left(\int^{\Lambda} \frac{d^3k}{(2\pi)^3} G_{\phi\phi}(k) + \frac{\langle \phi_1 \rangle^2}{N} \right) = m^2, \tag{32}$$

$$\langle \phi_1 \rangle = \frac{\sqrt{N} h^1}{m^2} , \tag{33}$$

where Λ is a cutoff on the maximum allowed $|k|$.

The third equation that completes the theory in equilibrium is

$$G_{\phi\phi}^{\mathrm{eq}}(k) = \frac{1}{k^2 + m^2} . \tag{34}$$

The critical point in the equilibrium theory is at $r_0 = r_c = -u\Lambda/2\pi^2$ and $h^{\alpha} = 0$. The two relevant operators that couple to r_0 and h^{α} are, respectively, $|\vec{\phi}|^2$ and ϕ_{α}, with scaling dimensions 1 and 5/2 in $d = 3$. They are, respectively, non-symmetry breaking and symmetry breaking. Consequently, we can study all three kinds of protocols along the r_0 axis ($\nu = 1, z = 2$) or along any of the h^{α} axes ($\nu = 2/5, z = 2$).

The equation of motion (25) for Model A dynamics is

$$\frac{\partial \phi_{\alpha}(\mathbf{k},t)}{\partial t} = -(k^2 + m^2)\phi_{\alpha}(\mathbf{k},t) + h^{\alpha}(2\pi)^3 \delta^3(\mathbf{k}) + \zeta_{\alpha}. \tag{35}$$

This specifies the functional dependence of $G_{\phi\phi}(k,t;\tau_{\varrho})$ on m^2 and completes the dynamical theory.

1. Scaling limit for TCPs and CCPs along the r_0 axis

Here, $h^{\alpha} = 0$ and $\delta(t;\tau_{\varrho}) = r_0(t;\tau_{\varrho}) - r_c$. Equation (33) implies that the one-point correlator for all components is zero: $\langle \phi_{\alpha} \rangle = 0$. We henceforth drop component subscripts. We will make frequent use of the Gaussian result (28), which we reproduce here for the reader's convenience:

$$G_{\phi\phi}(k,t;\tau_{\varrho};r_0) = 2 \int_{-\infty}^{t} dt' \, e^{-2 \int_{t'}^{t} dt'' [k^2 + r_0(t'';\tau_{\varrho})]}.$$

The notation $G_{\phi\phi}(k,t;\tau_{\varrho};r_0)$ emphasizes that $G_{\phi\phi}$ depends on $r_0(t;\tau_{\varrho})$.

Observe that the solution to Eq. (35) is exactly that of the free Gaussian case [Eq. (27)] with $r_0(t;\tau_{\varrho})$ replaced by $m^2(t;\tau_{\varrho})$. Consequently, the third equation that completes the

dynamical theory is

$$G_{\phi\phi}(k,t;\tau_\varrho) = G_{\phi\phi}(k,t;\tau_\varrho;m^2), \qquad (36)$$

where the right-hand side refers to the Green's function of (36) with $r_0(t;\tau_\varrho)$ replaced by $m^2(t;\tau_\varrho)$. The critical value r_c of r_0 in this notation is

$$r_c = -u \int^\Lambda \frac{d^3k}{(2\pi)^3} G_{\phi\phi}(k,t;\tau_\varrho;0). \qquad (37)$$

One can now re-express (32) as

$$m^2(t;\tau_\varrho) = \delta(t;\tau_\varrho) + u \int^\Lambda \frac{d^3k}{(2\pi)^3} [G_{\phi\phi}(k,t;\tau_\varrho;m^2) \\ - G_{\phi\phi}(k,t;\tau_\varrho;0)]. \qquad (38)$$

Let us now take the scaling limit. The KZ length and time scales are $t_\varrho = (\tau_\varrho^a/\delta_0)^{\frac{2}{2a+1}}$ and $l_\varrho = \sqrt{t_\varrho}$. Dimensional considerations imply that the scaling form of $m^2(t;\tau_\varrho)$ is $m^2(t;\tau_\varrho) \sim \mathcal{M}^2(\bar{t})/l_\varrho^2$. By using the leading-order behavior of the protocol near the critical point $\delta(t;\tau_\varrho) \sim (-\bar{t})^a/l_\varrho$, we derive the following scaled form of Eq. (38):

$$(-\bar{t})^a + 2u \int_{-\infty}^{\bar{t}} d\bar{t}' \int^\infty \frac{d^3\bar{k}}{(2\pi)^3} \\ \times \left(e^{-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}''(\bar{k}^2+\mathcal{M}^2)} - e^{-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}''\bar{k}^2} \right) = 0. \qquad (39)$$

We note the following points:

(1) The cutoff Λ appears in the unscaled equation only as the upper limit on the momentum integral. In the scaling limit, the upper limit $\sim \Lambda l_\varrho \to \infty$ and the scaled relation is *cutoff independent*.

(2) The scaled relation, being a limiting form, is *simpler* than the unscaled one.

(3) The relation above is unchanged by keeping higher-order terms in $r_0(t;\tau_\varrho)$. This is tantamount to proving the universality of all correlators with respect to details of the protocol away from the critical point in the scaling limit.

2. Scaling forms for the linear TCP

Fortunately, Eq. (39) can be solved when $a = 1$. The choice $u = \sqrt{8\pi}$ simplifies prefactors and makes apparent the form of the solution. Defining

$$f(\bar{t}) \equiv e^{2\int_0^{\bar{t}} d\bar{t}' \mathcal{M}^2(\bar{t}')} \quad \text{or} \quad \mathcal{M}^2(\bar{t}) = \frac{f'(\bar{t})}{2f(\bar{t})}, \qquad (40)$$

the solution is

$$f(\bar{t}) = -3^{1/3}\Gamma(1/3)e^{2\bar{t}^3/3}[\bar{t}\,\text{Ai}(\bar{t}^2) + \text{Ai}'(\bar{t}^2)].$$

Ai is the Airy function of the first kind and Γ is the gamma function. We may reduce $\mathcal{G}_{\phi\phi}$ to quadrature

$$\mathcal{G}_{\phi\phi}(\bar{k},\bar{t}) = 2 \int_{-\infty}^{\bar{t}} d\bar{t}' e^{-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}''[\bar{k}^2+\mathcal{M}^2(\bar{t}'')]},$$

and compute all higher cumulants using Wick's theorem.

As $\bar{t} \to -\infty$, we recover the equilibrium behavior in Eq. (31) with $\xi \sim 1/(-t)$. The results from the coarsening literature[36] in this theory are

$$l_{\text{co}}(t) = \sqrt{t}, \mathcal{G}_{\phi\phi}^{\text{co}}[kl_{\text{co}}(t)] = \exp\{-2[kl_{\text{co}}(t)]^2\}.$$
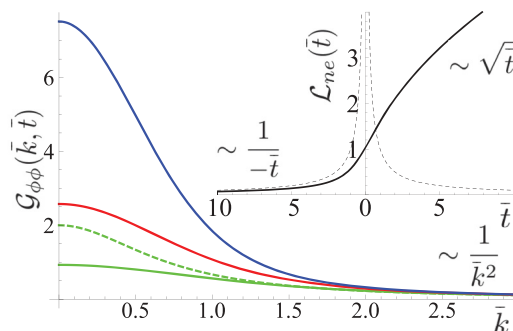

FIG. 4. (Color online) $\mathcal{G}_{\phi\phi}(\bar{k},\bar{t})$ vs $\bar{k}$ at fixed time slices for the linear TCP. The blue, red, and green solid lines are, respectively, at $\bar{t} = 0.5$, 0, and -0.5. The green dashed line is the correlator if the system were in equilibrium at $\bar{t} = -0.5$. Inset: $\mathcal{L}_{\text{ne}}$ vs $\bar{t}$ (solid line) and ξ/l_ϱ (dashed line).

We should therefore expect that $\mathcal{L}_{\text{ne}} \sim \sqrt{\bar{t}}$ and $\mathcal{G}_{\phi\phi} \sim \bar{t}^{5/2}\exp(-2\bar{k}^2\bar{t})$ as $\bar{t} \to \infty$. As is seen from Fig. 4, both asymptotes are correctly predicted for $\mathcal{L}_{\text{ne}}$. We have checked this for $\mathcal{G}_{\phi\phi}$ as well. This verifies that the asymptotes predicted in Eqs. (14) and (16) are correct in this theory.

3. Scaling limit for TCPs and CCPs along h₁ axis

Setting $r_0 = r_c$, we can explore symmetry-breaking protocols along the h^1 axis. The implicit relation from the self-consistency equations for the linear TCP is

$$\left(\int_{-\infty}^{\bar{t}} d\bar{t}' \,\bar{t}' \sqrt{f(\bar{t}')} \right)^2 = \int_{-\infty}^{\bar{t}} d\bar{t}' \frac{f(\bar{t}) - f(\bar{t}')}{|\bar{t} - \bar{t}'|^{3/2}}. \qquad (41)$$

Although scaling is guaranteed, we can not proceed further as the solution to this equation is not known.

V. GAUSSIAN QUANTUM FIELD THEORIES

We now turn to the KZ problem near quantum critical points. Again, it will prove instructive to investigate the class of ferromagnetic critical points with $O(N)$ symmetry. We will consider the simplest nontrivial case, that of Gaussian scalar fields. The Lagrangian we consider is

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 \right], \qquad (42)$$

where we permit m^2 to depend on time ($\delta \equiv m^2$). Standard critical exponents include $z = 1$ (owing to relativistic symmetry when m is a constant) and $\nu = 1/2$ (owing to the Yukawa form of the Green's function for static sources when m is a constant).

A. Second quantization

The second-quantized treatment of ϕ is based on the expansion

$$\phi(\mathbf{x},t) = \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_\mathbf{k}(t), \qquad (43)$$

where we set

$$\phi_\mathbf{k}(t) = f_\mathbf{k}(t)a_\mathbf{k} + f_{-\mathbf{k}}^*(t)a_{-\mathbf{k}}^\dagger \qquad (44)$$

and impose the commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}] = (2\pi)^d \delta^d(\mathbf{k} - \mathbf{k}'). \tag{45}$$

Owing to rotational symmetry, the mode functions $f_{\mathbf{k}}(t)$ and Green's functions only depend on the magnitude of the momentum k. The mode functions $f_k(t)$ satisfy the mode equation

$$\left[\frac{d^2}{dt^2} + \Omega_k^2(t) \right] f_k(t) = 0, \tag{46}$$

where

$$\Omega_k^2(t) \equiv k^2 + m^2(t). \tag{47}$$

One must also impose the Wronskian condition

$$f_k \dot{f}_k^* - \dot{f}_k f_k^* = i \tag{48}$$

in order to obtain standard commutation relations between $\phi_{\mathbf{k}}$ and its conjugate $\pi_{\mathbf{k}} = \dot{\phi}_{\mathbf{k}}^{\dagger}$. Once the mode functions f_k are specified, a Fock space vacuum $|0\rangle$ can be defined through the conditions $a_{\mathbf{k}}|0\rangle = 0$.

In general, Eq. (46) is hard to solve exactly. However, the techniques of Wentzel-Kramers-Brillouin (WKB) provide an approximate solution when Ω_k is real, positive, and varying "slowly enough"; to this end, let us define the oscillatory, positive frequency WKB solution

$$\tilde{f}_k(t) = \frac{1}{\sqrt{2\Omega_k(t)}} \exp\left\{ -i \int^t dt' \, \Omega_k(t') \right\}, \tag{49}$$

where the lower limit of integration can be specified at our later convenience. If one defines $\tilde{a}_{\mathbf{k}}(t)$ through the equations $\phi_k(t) = \tilde{f}_k(t)\tilde{a}_{\mathbf{k}}(t) + \tilde{f}_k^*(t)\tilde{a}_{-\mathbf{k}}^{\dagger}(t)$, and requires $[\tilde{a}_{\mathbf{k}}(t), \tilde{a}_{-\mathbf{k}}^{\dagger}(t)] = (2\pi)^d \delta^d(\mathbf{k} - \mathbf{k}')$ for all t, then $(\tilde{f}_k(t), \tilde{f}_k^*(t))$ and $(f_k(t), f_k^*(t))$ are related through the standard Bogoliubov transformation.

As m^2 is slowly varying and positive at large negative times, the evolution is adiabatic and we require that the f_k coincide asymptotically with the positive frequency WKB solutions $\tilde{f}_k$ as $t \to -\infty$. Thus, $|0\rangle$ is the "out" vacuum in the parlance of Refs. 52 and 53, and we assume that our system has been prepared in this vacuum. All expectation values unless otherwise indicated are with respect to this state.

B. Scaling limit

In the classical context, we first solved the complete dynamical problem for arbitrary initial condition and protocol choice. We then established that all local physical quantities reduced to universal forms *independent of the details of the protocol away from the critical point and the initial conditions* in the KZ scaling limit. We repeat the same exercise here by using WKB methods to solve Eq. (46) for a general TCP/CCP in $m^2(t; \tau_\varrho)$.

Assume that the system is initially prepared in the vacuum state. The solution to Eq. (46) is well approximated by the positive oscillatory WKB solution $\tilde{f}_k$ at early times, and by a linear combination of the WKB solutions $\tilde{f}_k$ and $\tilde{f}_k^*$ at late times, as long as the frequency Ω_k is much larger than the rate at which the frequency changes $|d\Omega_k/dt|$. This yields the condition $|dm^{-1}/dt| \ll 1$. Within the window $|t| < \tau_\varrho$,

we use the leading-order expansion for $m^2(t; \tau_\varrho)$ in t/τ_ϱ,

$$m^2(t; \tau_\varrho) = m_0^2 \left(-\frac{t}{\tau_\varrho} \right)^a \left[1 + a_1 \frac{t}{\tau_\varrho} + \cdots \right], \tag{50}$$

to conclude that the WKB solutions are valid as long as $|t| \gg t_\varrho$ or $|\bar{t}| \gg 1$. Note that t_ϱ using Eq. (5) is

$$t_\varrho = \tau_\varrho^{\frac{a}{2+a}} \Big/ m_0^{\frac{2}{2+a}}. \tag{51}$$

Directly solving the mode equation using the expansion in Eq. (50) yields a solution valid in the region $|t| \ll \tau_\varrho$ or $|\bar{t}| \ll \tau_\varrho/t_\varrho$. Thus, the overlap in the ranges of the validity of the direct solution and the WKB one is

$$1 \ll \bar{t} \ll \frac{\tau_\varrho}{t_\varrho}. \tag{52}$$

We are now in a position to take the KZ scaling limit whereby $\tau_\varrho \to \infty$ and all quantities are measured in units of the KZ length and time. We first notice that the region of overlap in Eq. (52) diverges. The mode equation near $t = 0$ also simplifies in scaled units. We see from Eq. (48) that f_k carries dimensions of $\sqrt{t}$, so its scaling form is

$$f_k \sim \sqrt{t_\varrho} \, \bar{f}_{\bar{k}}, \tag{53}$$

where $\bar{k} = l_\varrho k$ as usual and $l_\varrho = t_\varrho$. The scaled mode equation in the window $|t| < \tau_\varrho$ before any limits are taken is

$$\left[\frac{d^2}{d\bar{t}^2} + \bar{\Omega}_{\bar{k}}^2(\bar{t}) \right] \bar{f}_{\bar{k}} = 0, \tag{54}$$

where $\bar{\Omega}_{\bar{k}}^2 = \bar{k}^2 + \bar{m}^2(\bar{t})$ and

$$\bar{m}^2(\bar{t}) = (-\bar{t})^a \left[1 + a_1 \frac{t_\varrho \bar{t}}{\tau_\varrho} + a_2 \left(\frac{t_\varrho \bar{t}}{\tau_\varrho} \right)^2 + \cdots \right]. \tag{55}$$

In the KZ scaling limit, $t_\varrho/\tau_\varrho \to 0$ and the corrections in the square brackets vanish, resulting in the simpler scaled mode equation

$$\left[\frac{d^2}{d\bar{t}^2} + \bar{k}^2 + (-\bar{t})^a \right] \bar{f}_{\bar{k}} = 0. \tag{56}$$

The goodness of the approximation is parametrically controlled by the smallness of the parameter t_ϱ/τ_ϱ. Corrections to the diverging overlap in the ranges of the validity of the solution to the scaled equation above and the WKB one is also controlled by the same small parameter t_ϱ/τ_ϱ. Thus, the solution to Eq. (54) picked out by starting with a positive frequency WKB solution at early times is the same as the one picked out by applying the early-time positive frequency condition directly to the solutions of the limiting equation (56). This is the sense in which the mode functions in the scaling limit are universal. It is worth noting that these considerations are merely an elaboration of the standard arguments used to justify turning-point formulas in standard WKB treatments of the time-independent Schrödinger equation in the geometric optics limit.

Finally, let us note that the above considerations will not apply to the marginal ECP. For a given nonuniversal regularization of the protocol at small time ($t < \tau_\varrho$), the direct solution to the mode equation in the power-law regime ($t \gg \tau_\varrho$) has no overlap with the positive-frequency WKB solution at early times. As we discuss later, this implies that

some information about the short-time regularization *must* enter the scaling limit.

C. Quasiparticles, heat density, and diagonal entropy

As the Gaussian problem is integrable, quasiparticles are well defined and infinitely long lived. The quasiparticle number at momentum **k**, $\mathcal{N}_k$, is defined as the expectation value of the occupation of mode **k** and has the scaling form

$$\mathcal{N}_{\bar{k}}(\bar{t}) \equiv \langle \tilde{a}_{\mathbf{k}}^{\dagger}(t) \tilde{a}_{\mathbf{k}}(t) \rangle$$
$$= \frac{1}{2|\bar{\Omega}_{\bar{k}}|} |\partial_{\bar{t}} \bar{f}_{\bar{k}} + i \bar{\Omega}_{\bar{k}} \bar{f}_{\bar{k}}|^2. \tag{57}$$

The excess energy density is given by $q(t; \tau_\varrho) = \int d^d k/(2\pi)^d \Omega_k \mathcal{N}_k$ and is easily seen to obey the scaling form conjectured in Eq. (23). A definition of the entropy density of the system at each instant of time is through the diagonal entropy density. The diagonal entropy density s is the entropy density of the diagonal components of the density matrix $\rho(t; \tau_\varrho)$ in the many-body adiabatic basis. In the Gaussian problem, s is additive in the label **k** and this definition simplifies to

$$s(t; \tau_\varrho) = - \int \frac{d^d k}{(2\pi)^d} \sum_m \rho_{m\mathbf{k}, m\mathbf{k}} \ln(\rho_{m\mathbf{k}, m\mathbf{k}}).$$

Here, $\rho_{m\mathbf{k}, m\mathbf{k}}$ is the absolute value squared of the overlap between the time-evolved wave function $|\psi(t)\rangle$, and the mth excited state of the harmonic oscillator labeled by **k**. The integral only runs over half the volume in k space as the modes **k** and $-\mathbf{k}$ are coupled by the Hamiltonian. $\rho_{m\mathbf{k}, m\mathbf{k}}$ can be directly computed in the Schrödinger picture because the time-evolved wave function is known in terms of the mode functions. In the eigenbasis of the operator $\phi_{\mathbf{k}}$ defined in Eq. (44), the wave function is a Gaussian[54]

$$\langle \phi_k | \psi \rangle \propto \exp\left[- \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{|f_k(t)|^2} - i \frac{\dot{f}_k(t)}{f_k(t)} \right) \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \right]$$

up to normalization and time-dependent phases. In the scaling limit, $f_k(t) \sim \sqrt{t_\varrho} \bar{f}_{\bar{k}}(\bar{t})$ and s has the scaling form predicted in Eq. (23). As all that is at issue is the scaling of the mode functions $f_k(t)$, the scaling forms of correlation functions may also be easily verified:

$$G_{\phi\phi}(k, t; \tau_\varrho) \equiv |f_k(t)|^2 \sim t_\varrho |\bar{f}_{\bar{k}}(\bar{t})|^2 \sim t_\varrho \mathcal{G}_{\phi\phi}(\bar{k}, \bar{t}). \tag{58}$$

D. Linear protocol

We now turn to a particular protocol, namely, the linear quench, where the KZ time is $t_\varrho = (\tau_\varrho/m_0^2)^{1/3}$. For this case, the mode equation can be solved in closed form to give

$$\bar{f}_{\bar{k}}(\bar{t}) = \sqrt{\frac{\pi}{2}} [\mathrm{Bi}(\bar{t} - \bar{k}^2) + i \, \mathrm{Ai}(\bar{t} - \bar{k}^2)], \tag{59}$$

where Ai and Bi are Airy functions of the first and second kinds. The scaling function of the two-point function from Eq. (58) is

$$\mathcal{G}_{\phi\phi}(\bar{k}, \bar{t}) = \frac{\pi}{2} [\mathrm{Ai}^2(\bar{t} - \bar{k}^2) + \mathrm{Bi}^2(\bar{t} - \bar{k}^2)]. \tag{60}$$
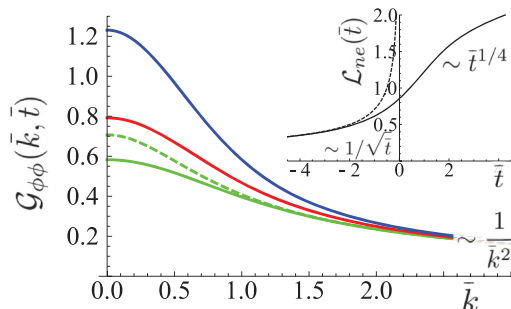
FIG. 5. (Color online) $G_{\phi\phi}$ for the linear TCP. The blue, red, and green solid lines are, respectively, at $\bar{t} = 0.5$, 0, and -0.5. The green dashed line is the correlator if the system were in equilibrium at $\bar{t} = -0.5$. Inset: $\mathcal{L}_{\mathrm{ne}}$ vs $\bar{t}$ (solid line) and ξ/l_ϱ (dashed line).

As a check, we may retrieve the equilibrium result when $\hat{t} \to -\infty$ holding $k\xi$ or $\bar{k}/\sqrt{-\bar{t}}$ fixed:

$$\mathcal{G}_{\phi\phi}(\bar{k}, \bar{t}) \sim \frac{1}{2\bar{\Omega}_{\bar{k}}}. \tag{61}$$

The equilibrium result is not retrieved when $\bar{t} \to \infty$ because of the pathology of the inverted ϕ^2 term when $\bar{t} > 0$. Instead, $\mathcal{G}_{\phi\phi}$ grows exponentially with $\bar{t}$. From $G_{\phi\phi}$ we can calculate the nonequilibrium correlation length

$$\xi_{\mathrm{ne}} \sim t_\varrho \sqrt{\frac{\partial_{\bar{t}}[\mathrm{Ai}(\bar{t})^2 + \mathrm{Bi}(\bar{t})^2]}{\mathrm{Ai}(\bar{t})^2 + \mathrm{Bi}(\bar{t})^2}}, \tag{62}$$

the scaling form $\mathcal{L}_{\mathrm{ne}}$ of which is plotted in Fig. 5(a). At large positive t, $\mathcal{L}_{\mathrm{ne}}$ grows polynomially instead of exponentially with time, despite the instability of the adiabatic theory with an inverted ϕ^2 term. As the adiabatic problem is pathological for $\bar{t} > 0$, we can only sensibly talk of quasiparticle occupations and thermodynamic quantities as long as $\bar{t} \leqslant 0$. In Fig. 6, we plot $\mathcal{N}_{\bar{k}}(\bar{t})$ and $\mathcal{S}_{\bar{k}}(\bar{t})$ for various values of $\bar{k}$ (the behavior of the excess energy density can be inferred from the number of quasiparticle excitations). The contribution to the total quasiparticle density from the high-wave-number ($\bar{k} \gg \sqrt{-\bar{t}}$) modes for $\bar{t} \ll -1$ is finite only if $d < 6$:

$$\int_{\sqrt{-\bar{t}}}^{\infty} \frac{d^d \bar{k}}{(2\pi)^d} \mathcal{N}_{\bar{k}}(\bar{t}) \sim \frac{1}{(-\bar{t})^{3-d/2}} \quad \text{if} \quad d < 6.$$

The total quasiparticle density and, consequently, the energy and entropy densities, are ill defined in the scaling limit within this integrable model when $d \geqslant 6$. Additionally, observe that $\mathcal{N}_{\bar{0}}$ diverges at $\bar{t} = 0$ because the gap Ω_0 between the ground and excited states of the oscillator at $\bar{k} = 0$ closes at $\bar{t} = 0$.
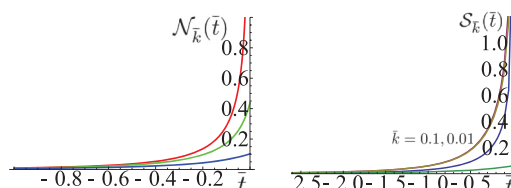
FIG. 6. (Color online) Left: Quasiparticle number $\mathcal{N}_{\bar{k}}(\bar{t})$ for $\bar{k} = 0, 0.25, 0.5$. Right: Diagonal entropy $\mathcal{S}_{\bar{k}}$ for $\bar{k} = 0, 0.01, 0.1, 1$ from top to bottom in the linear TCP.

E. CCP: Quadratic protocol

We now turn to a case where the adiabatic problem is well defined for all $\bar{t}$: the quadratic CCP. The KZ time here is $t_Q = (\tau_Q/m_0)^{1/2}$. The scaling forms of the mode functions and the equal-time two-point function are

$$\bar{f}_{\bar{k}}(\bar{t}) = 2^{-1/4} e^{-\frac{\pi \bar{k}^2}{8}} D_{\frac{-1+i\bar{k}^2}{2}}\left(-\sqrt{2} e^{-\frac{i\pi}{4}} \bar{t}\right),$$

$$\mathcal{G}_{\phi\phi}(\bar{k},\bar{t}) = \frac{e^{-\frac{\pi\bar{k}^2}{4}}}{\sqrt{2}} \left| D_{\frac{-1+i\bar{k}^2}{2}}\left(-\sqrt{2} e^{-\frac{i\pi}{4}} \bar{t}\right)\right|^2. \quad (63)$$

$D_\nu(z)$ is the parabolic cylinder function.

As $\bar{t} \to \pm\infty$ holding $k\xi$ or $\bar{k}/\bar{t}$ fixed, we recover the equilibrium forms

$$\mathcal{G}_{\phi\phi}(\bar{k},\bar{t}) \sim \frac{1}{2\bar{\Omega}_{\bar{k}}}. \quad (64)$$

The retrieval of the equilibrium form as $\bar{t} \to \infty$ is by no means guaranteed. Recent work[30] suggests that this question is intimately tied to the dephasing of the off-diagonal terms in the time-averaged density matrix in the instantaneous eigenbasis. If the off-diagonal terms do not dephase, which would be the case for example for the quartic quench, then equilibrium behavior is not expected as $\bar{t} \to \infty$.

The Gaussian theory imposes further structure on correlators because each momentum mode evolves independently of the others. Consider the time evolution of the wave function of the harmonic oscillator labeled by $\bar{k}$ in the Schrödinger picture. Expanding it in the eigenbasis of the Hamiltonian at time $\bar{t}$ and suppressing the label $\bar{k}$,

$$|\psi\rangle \approx A_0 |0(\bar{t})\rangle + A_1 |1(\bar{t})\rangle \dots.$$

The time evolution is adiabatic when $\bar{t} \gg 1$ and each $|A_i|$ approaches a constant. The relative phase between A_{i+1} and A_i on the other hand grows as $\int^{\bar{t}} \bar{\Omega}_{\bar{k}}(\bar{t}')d\bar{t}' \sim \bar{t}^2$. This is the origin of the oscillations of period $1/\bar{t}$ at large positive $\bar{t}$ at each $\bar{k}$, seen, for example, in $\mathcal{L}_{ne}$ in Fig. 7(b).

The simple definition of ξ_{ne} in Sec. III A 4 has to be modified as $\mathcal{G}_{\phi\phi}(\bar{k},\bar{t})$ has multiple poles in the complex $\bar{k}$ plane for late times. The pole with the smallest imaginary part determines the decay constant $(1/\xi_{ne})$ over the longest length scales in the two-point function. The nonequilibrium correlation length is plotted in Fig. 7(b). As $\bar{t} \to \pm\infty$, the envelope of ξ_{ne} behaves as
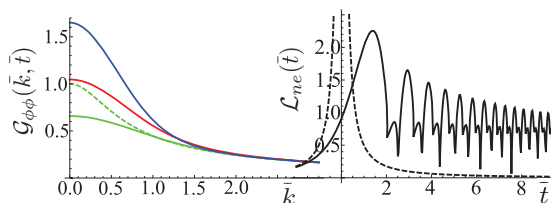
$$\xi_{ne}(\bar{t}) \sim \frac{1}{|\bar{t}|}.$$

FIG. 7. (Color online) Left: $G_{\phi\phi}$ for the quadratic CCP. The blue, red, and green solid lines are, respectively, at $\bar{t} = 0.5, 0,$ and -0.5. The green dashed line is the correlator if the system were in equilibrium at $\bar{t} = -0.5$. Right: $\mathcal{L}_{ne}$ vs $\bar{t}$ (solid line) and ξ/l_Q (dashed line).
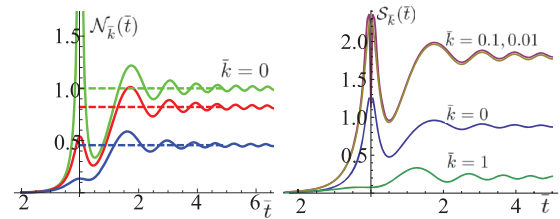

FIG. 8. (Color online) Left: Quasiparticle number $\mathcal{N}_{\bar{k}}(\bar{t})$ for $\bar{k} = 0, 0.25, 0.5$. Right: Diagonal entropy $\mathcal{S}_{\bar{k}}$ for $\bar{k} = 0, 0.01, 0.1, 1$ from top to bottom in the quadratic CCP.

This confirms the prediction in Eq. (16) for CCPs with no coarsening physics.

Finally, we plot $\mathcal{N}_{\bar{k}}(\bar{t})$ and $\mathcal{S}_{\bar{k}}(\bar{t})$ for various values of $\bar{k}$ in Fig. 8. In each oscillator, the quasiparticles and entropy are essentially produced in the time interval $|\bar{t}| \lesssim 1$. At late times, both quantities settle to a constant dependent on $\bar{k}$. For example,

$$\mathcal{N}_{\bar{k}}(\bar{t}) \to e^{-\pi\bar{k}^2} \quad \text{as} \quad \bar{t} \to \infty.$$

As was the case for the linear quench, the total quasiparticle and entropy densities are finite only if $d < 6$.

F. Marginal ECP

The relativistic Gaussian theory provides an ideal setting to study the nonequilibrium states that arise in the marginal ECP. The protocol is defined by Eq. (21): $m(t)$ smoothly transitions from a constant to θ/t over the time scale τ_Q and $\dot{\xi}_t$, the parameter that controls adiabaticity, equals $1/\theta$ for $t \gg \tau_Q$. Our naive expectation is that the mode function is of a universal scaling form $f_k(t) = \sqrt{t}\,\bar{f}(kt)$, in the scaling limit $t \to \infty, k \to 0$ with kt held fixed.

This expectation is violated on two fronts even in the Gaussian problem. The first is that the marginal ECP is by construction unable to entirely "forget" its early-time regularization. More precisely, observe that our choice of protocols implies that $|dm^{-1}/dt| \ll 1$ when $t \ll \tau_Q$ while $|dm^{-1}/dt| = 1/\theta$ when $t \gg \tau_Q$. The latter result shows that for small enough k, it is not possible to reach the power-law regime while remaining adiabatic, in contrast to our discussion of the TCP and CCP cases; thus, we should expect that *some* nonuniversal information must make its way into the putative scaling regime. The second and more striking violation is that of scaling. When $\theta > 1/2$ and $|dm^{-1}/dt|$ is closer to obeying the condition of adiabaticity, scaling is violated "mildly" and all physical quantities are periodic functions of $\ln(t)$. When $\theta = 1/2$, scaling is violated logarithmically. The most dramatic violation of scaling is when $0 < \theta < 1/2$. Here, the adiabaticity condition is strongly broken and the scalar field acquires an anomalous dimension.

Let us now describe how these features emerge in the long-time, small momentum form of the mode functions. The mode equation for $t \gg \tau_Q$

$$\left(\frac{d^2}{dt^2} + k^2 + \frac{\theta^2}{t^2}\right) f_k(t) = 0. \quad (65)$$

Define $\lambda \equiv |\sqrt{\frac{1}{4} - \theta^2}|$ and

$$h(\lambda) = \begin{cases} i\lambda & \text{if} \quad \theta > 1/2, \\ 0 & \text{if} \quad \theta = 1/2, \\ \lambda & \text{if} \quad \theta < 1/2. \end{cases} \tag{66}$$

Setting $k = 0$, Eq. (65) is solved as

$$f_0(t) = \begin{cases} \sqrt{t}\left[\frac{2^{-\lambda}u}{\Gamma(1+\lambda)}\left(\frac{t}{\tau_\varrho}\right)^{h(\lambda)} + \frac{-2^\lambda \Gamma(\lambda)v}{\pi}\left(\frac{t}{\tau_\varrho}\right)^{-h(\lambda)}\right] \\ \text{if} \quad \theta \neq 1/2, \\ \sqrt{t}\left[u + v\ln\left(\frac{t}{\tau_\varrho}\right)\right] \quad \text{if} \quad \theta = 1/2. \end{cases}$$

The units of t are chosen such that $f_0(\tau_\varrho) \sim \sqrt{\tau_\varrho}$. The particular λ-dependent prefactors are chosen to simplify the solution at all k. v and u are nonuniversal constants in the equation above. They depend on the detailed preasymptotic form of the protocol and are fixed by the solution to the differential equation with the full time dependence of the mass.

Let us now turn to $k \neq 0$. For times greater than τ_ϱ, the mode equation is solved by the linear combination

$$f_k(t) = \sqrt{t}[c_J(k) J_{h(\lambda)}(kt) + c_Y(k) Y_{h(\lambda)}(kt)]. \tag{67}$$

$J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of order ν of the first and the second kind. The functions $c_J(k)$ and $c_Y(k)$ are presently undetermined except for the requirement of smoothness in k. However, if we examine the region $t \gg \tau_\varrho$ at small kt, we can obtain their leading-order behavior by requiring that they match smoothly to the $k = 0$ forms presented earlier. This localizes the nonuniversality to the constants u, v discussed already. For $\theta < 1/2$ and k smaller than some nonuniversal k_0, this leads to the leading behavior

$$c_Y(k) = v(k\tau_\varrho)^\lambda, \quad c_J(k) = u(k\tau_\varrho)^{-\lambda} - v\frac{(k\tau_\varrho)^\lambda}{\pi}\cot\pi\lambda. \tag{68}$$

For $\theta > 1/2$, we simply replace λ by $i\lambda$. For $\theta = 1/2$, we find

$$c_Y(k) = \frac{\pi}{2}v, \quad c_J(k) = u - v\ln\frac{k\tau_\varrho e^\gamma}{2}. \tag{69}$$

γ is the Euler-Mascheroni constant. We emphasize that the mode function and consequently all physical quantities are well approximated by Eq. (67) with the forms of c_J and c_Y above for $t \gg \tau_\varrho$ and $k \ll k_0$.

We now turn to the physical implications of these solutions for an arbitrary choice of (u, v) that satisfy the constraint imposed by the Wronskian condition (48). First consider the case $0 < \theta < 1/2$, which is strongly nonadiabatic in the sense that $|dm^{-1}/dt| = 1/\theta$ is greater than 2 and may be large. The mode function in Eq. (67) can be rewritten in the form

$$f_k(t) = t^{1/2}\left(\frac{t}{\tau_\varrho}\right)^\lambda\left[g_+(kt) + \left(\frac{t}{\tau_\varrho}\right)^{-2\lambda} g_-(kt)\right]. \tag{70}$$

g_+ and g_- are linear combinations of the two Bessel functions and involve the constants u, v. Observe that in the limit of large t ($t \gg \tau_\varrho$) with kt fixed, the ratio of the second to the first term in the expression above decreases as $(t/\tau_\varrho)^{-2\lambda}$.[55] Thus, in the scaling limit, $f_k(t)$ simplifies to

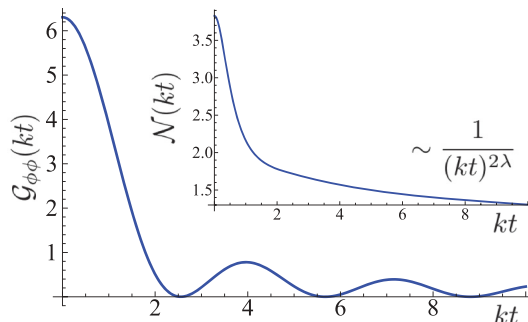$$f_k(t) \sim u' t^{1/2+\lambda} \bar{f}(kt) \sim u' t^{1/2+\lambda}\frac{J_\lambda(kt)}{(kt)^\lambda}. \tag{71}$$

FIG. 9. (Color online) $\mathcal{G}_{\phi\phi}(kt)$ and $\mathcal{N}(kt)$ vs kt when $\lambda = 0.1$ and $u/\tau_\varrho^\lambda = -2.5i$.

$u' = u/\tau_\varrho^\lambda$ above. The scaling forms predicted in Sec. III B *do not hold*. Instead, the form above is the one expected when the field ϕ has an anomalous dimension λ. The scaling form of $G_{\phi\phi}$ with the modified dimension of ϕ is

$$G_{\phi\phi}(k,t) \sim t^{1+2\lambda}\mathcal{G}_{\phi\phi}(kt). \tag{72}$$

In real space, this implies that the equal-time two-point correlator decays as $1/x^{d-1-2\lambda}$ at fixed x/t. Analogously, the quasiparticle number $N_k(t)$ has the scaling form $t^{2\lambda}\mathcal{N}(kt)$. The scaling functions are

$$\mathcal{G}_{\phi\phi}(kt) = |u'|^2 \bar{f}^2,$$

$$\mathcal{N}(kt) = |u'|^2 \frac{|(kt)\partial_{kt}\bar{f} + i\sqrt{(kt)^2 + (\theta)^2}\,\bar{f}|^2}{2\sqrt{(kt)^2 + (\theta)^2}}.$$

They are shown in Fig. 9. Three comments are in order. First, in the marginal ECP, $\xi \sim t$ and the time scale for a change in m is the same as $\xi = \xi_t$. We therefore expect that the nonequilibrium correlation length ξ_{ne} also grow linearly in time. This is indeed the case; the oscillations in $\mathcal{G}_{\phi\phi}(kt) \sim t$ in Fig. 9 are of order one period and indicate a peak at $r \sim t$ in the real-space correlator $\mathcal{G}_{\phi\phi}(r/t)$. Second, the excess energy density above the instantaneous vacuum *decreases* as $1/t^{1-2\lambda}$. The marginal ECP thus leads the system to the critical point through a family of new, nonequilibrium states. Finally, all scaling functions are known up to a multiplicative constant (u') in the scaling limit.

Let us consider the case $\theta > 1/2$, which is weakly nonadiabatic in the sense that $|dm^{-1}/dt| = 1/\theta$ is less than 2 and may be small. A rewriting of the mode function informs us that the scaling is violated only by phases:

$$f_k(t) = \sqrt{t}\left[\left(\frac{t}{\tau_\varrho}\right)^{i|\lambda|} g_+(kt) + \left(\frac{t}{\tau_\varrho}\right)^{-i|\lambda|} g_-(kt)\right]. \tag{73}$$

These phases affect other physical quantities in the scaling limit, but in relatively mild ways as compared to the previously seen factors of t^λ. For instance, at late times holding kt fixed,

$$G_{\phi\phi}(k,t) \sim t\, \mathcal{G}_{\phi\phi}(kt, e^{i|\lambda|\ln(t/\tau_\varrho)}), \tag{74}$$

where we have expressed $t^{i|\lambda|} = e^{i|\lambda|\ln t}$ in order to emphasize that this quantity is periodic in $\ln t$. The scaling function $\mathcal{N}_k$ is also periodic in $\ln t$. The two "almost-scaling" functions $\mathcal{G}_{\phi\phi}$ and $\mathcal{N}_k(t)$ are plotted in Fig. 10 for three equally spaced values of $\ln t$ within a decade. The late-time behavior of $\mathcal{G}_{\phi\phi}$ in the marginal ECP differs markedly from the adiabatic response,
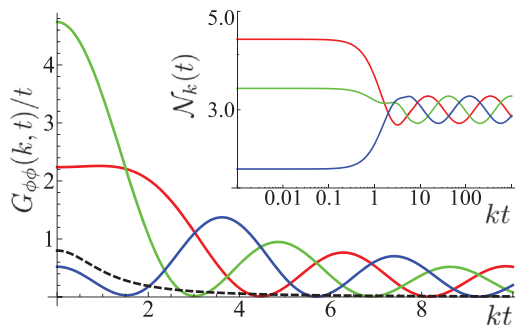
FIG. 10. (Color online) $G_{\phi\phi}(k,t)/t$ and $\mathcal{N}_k(t)$ vs kt when $\lambda = i$ plotted at three equally spaced time on the $\ln(t)$ scale: $t = 10$ (red line), $10 e^{\pi|\lambda|/3}$ (green line), and $10 e^{2\pi|\lambda|/3}$ (blue line). The dashed line is the adiabatic response. (u,v,τ_ϱ) are chosen to be $(\sqrt{3/2},1,1)$.

shown as a dashed line in the plot. The period oscillations in the two-point correlator imply that $\xi_{\text{ne}} \sim t$. The smaller excess energy density proportional to $1/t$ is an indicator that the system is closer to being in equilibrium than for the strongly nonadiabatic case $0 < \theta < 1/2$. In a sense, the factors of $t^{i|\lambda|} = e^{i|\lambda|\ln t}$ can be regarded as introducing only "logarithmic" modifications of scaling.

The case $\theta = 1/2$ displays the properties of both the cases discussed above. The relations in Eq. (69) imply the form

$$f_k(t) = \sqrt{t}[g_+(kt) + \ln(t/\tau_\varrho)g_-(kt)].$$

At late times with kt fixed, the second term dominates the first and the mode function has the scaling form

$$f_k(t) \sim u \sqrt{t} \ln t \, J_0(kt).$$

All scaling forms are thus modified by prefactors of powers of $\ln t$. The logarithmic violation of scaling is similar to the weakly nonadiabatic case $\theta > 1/2$, while the modification of the dimension of ϕ is similar to the strongly nonadiabatic case. The quasiparticle number here diverges logarithmically in time.

The excess energy density injected into the system decreases as t increases in the strictly Gaussian theory. The marginal ECPs thus define a family of nonequilibrium states that lead to the critical point that are universal when the Gaussian fixed point is stable. The vanishing excess energy density further suggests that the nonequilibrium states generated by the marginal ECP survive even when the fixed point is not Gaussian.

Finally, let us comment on the ECP in the transverse-field Ising model in $(1 + 1)$ dimensions, which is famously a model of free fermions. Here, $\nu = 1$ and $z = 1$ so the relevant power is now $a = 1$. The explicit solution of the fermionic mode equations exhibits the analogs of our weakly nonadiabatic regime with corrections to scaling that are periodic in $\ln t$.[32] Connecting this behavior to our Gaussian results discussed above by continuation in the number of dimensions is an interesting challenge for future work.

1. Analogies to dS_{d+1}/CFT_d

We showed above that in the long-time limit of the solution to the mode equation in the strongly nonadiabatic case Eq. (70),

the scalar field acquires an anomalous dimension λ. If we instead took the opposite limit of small time ($t \sim \tau_\varrho$), the second term is more important than the first and the anomalous dimension of the scalar field is $-\lambda$. Thus, the effective scaling dimension of ϕ in dimension d is

$$\tilde{\Delta}_\pm = \frac{d-1}{2} \pm \lambda \tag{75}$$

at short and long times, respectively. Readers familiar with the $(A)dS_{d+1}/(C)FT_d$ literature, in particular,[56,57] will note a similarity between the result Eq. (72) and the d-dimensional field-theory Green's function obtained in the presence of a double-trace deformation. We can make the analogy closer by noting that the modified scalar field

$$\varphi \equiv t^{\frac{d-1}{2}}\phi$$

obeys an equation of motion which follows from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\sqrt{-\det g_{\alpha\beta}}(-g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - M^2\varphi^2), \tag{76}$$

where $g_{\mu\nu}$ is the metric of de Sitter space, dS_{d+1}:

$$ds^2 = \frac{L^2}{t^2}(-dt^2 + d\vec{x}^2). \tag{77}$$

The length scale L is arbitrary. The de Sitter mass must be given by

$$M^2 L^2 = \theta^2 + \frac{d^2-1}{4} \tag{78}$$

in order for the equation of motion from Eq. (77) to agree with Eq. (65). The dimensions Eq. (75) are closely related to the usual ones in dS/CFT:

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - M^2 L^2} = \tilde{\Delta}_\pm + \frac{1}{2}. \tag{79}$$

The difference arises due to an important distinction between the late-time Green's function (72) and the d-dimensional field-theory two-point function. Examples of the latter could be computed in dS_{d+1} in terms of early time (that is, small t) properties of mode functions associated to the "out" vacuum: that is, mode functions which are purely positive frequency at late times. Our computation is the reverse of this, in the sense that we investigate late-time properties of mode functions associated with an "in" vacuum. Late time corresponds to the deep interior of dS_{d+1} (more precisely, it is a corner of the global covering space far from the boundary at $t = 0$). Another difference (of lesser consequence) is that in the normal parlance of dS/CFT,[56] time flow would be reversed, so that what we call $t = 0$ is the far future, while $t \to \infty$ is the far past.

In the window $kt \ll 1$, $f_k(t)$ is k independent and equal to $f_0(t)$:

$$t^{\frac{d-1}{2}} f_k(t) \approx a_\varphi t^{\Delta_-} + b_\varphi t^{\Delta_+}, \tag{80}$$

where b_φ, a_φ are known in terms of u, v, τ_ϱ, and in particular have a definite ratio. This setup now bears a strong resemblance to double-trace operator deformations in AdS/CFT.[57] In AdS/CFT, the relativistic conformal symmetry of the boundary theory is broken by the multitrace deformation. The presence of different powers of x in the real-space

Green's function dependent on the energy scale $1/\tau_\varrho$ signals a similar breakdown of scaling. The same can be said of the weakly nonadiabatic case, but comparisons with a boundary field theory are harder in this case because the field theory would have to be nonunitary, similar to violations of the Breitenlohner-Freedman bound[58] in AdS/CFT. However, the dS_{d+1} formulation (76) does offer some further intuition regarding the weakly nonadiabatic case: Eq. (78) shows that large θ implies large M^2L^2. This in turn implies that the Compton wavelength of the massive scalar is much smaller than the Hubble scale of dS_{d+1}, which is precisely the condition one needs in order to justify a geometric optics approximation.

VI. CLOSING REMARKS

Our primary purpose in this paper has been to systematize the universal content of the KZ problem and to emphasize that all physical quantities give rise to universal scaling functions that span the entire crossover from equilibrium at early times to the late-time state. We have presented model computations that bear out this logic. Experiments directed towards observing this broader scaling picture would be highly desirable. We note that the scaling ideas presented here do not rely on the existence of a local order parameter and generalize straightforwardly to the Rajanti-Hindmarsh mechanism[59] as we will discuss elsewhere.

An obvious challenge is to extend such computations to more physically realizable problems where the field theories are not as simple. We will present some results on nontrivial, but not physically realizable field theories, via the AdS/CFT correspondence elsewhere. Another obvious challenge is to formulate a renormalization group procedure that makes the universality manifest, beyond the case of stochastic classical models with the associated functional integral formalism discussed here.

A by-product of our analysis has been the identification of an especially interesting ECP which is able to produce anomalous dimensions already at the Gaussian level, through a mechanism similar to the way anomalous dimensions emerge in (A)dS/CFT. A deeper understanding of this phenomenon and its examination in interacting contexts is also a fit subject for further exploration.

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APPENDIX: SCALING FORM OF $f(t; \tau_\varrho)$ IN THE GAUSSIAN THEORY WITH MODEL A DYNAMICS

To determine the generating functional in one higher dimension, we follow the three steps prescribed in Refs. 42 and 43:

(i) Define ϕ_ζ as the solution of the equation of motion in Eq. (25) for a given noise history ζ: $\mathcal{J}\phi_\zeta(\mathbf{k},t) = \zeta(\mathbf{k},t)$. $\mathcal{J}$ is a linear operator in the Gaussian theory.

(ii) Rewrite Z as $\int d\zeta\, P(\zeta) \exp(J\phi_\zeta)$. The noise distribution $P(\zeta)$ is Gaussian.

(iii) Recognize that the probability distribution for ϕ_ζ is related to the noise distribution as $P_\phi(\phi_\zeta) = P(\mathcal{J}\phi_\zeta)\det(\mathcal{J})$.

The generating functional of correlation functions of ϕ so obtained is

$$Z[J, \tau_\varrho] = \int \mathcal{D}\phi \, \det(\mathcal{J}) \, e^{\int d^3k \, dt \, [-2|\mathcal{J}\phi(\mathbf{k},t)|^2 + J(\mathbf{k},t)\phi(-\mathbf{k},t)]},$$

(A1)

where

$$\mathcal{J} = \frac{\partial}{\partial t} + k^2 + r_0(t; \tau_\varrho). \tag{A2}$$

The functional integral is Gaussian, and the free-energy density is expressed in terms of the structure factor in Eq. (28) as

$$f(t; \tau_\varrho) = \int \frac{d^3k}{(2\pi)^3} \ln\left[G_{\phi\phi}^{-1}(k, t; \tau_\varrho)\right].$$

The challenge in identifying f_{na}, even in the time-independent setting, lies in subtracting cutoff-dependent terms from f that are analytic in δ. Here, it involves subtracting the cutoff-dependent equilibrium contribution at the critical point $f(0, \infty)$, and two terms that are linear and logarithmic in the cutoff. On taking the KZ scaling limit of the terms remaining, we confirm the scaling form in Eq. (19):

$$f_{\text{na}}(t; \tau_\varrho) \sim \frac{1}{l_\varrho^d}\mathcal{F}(\bar{t}),$$

$$\mathcal{F}(\bar{t}) = \frac{1}{6\pi^2}\int_0^\infty d\bar{k}\left[\bar{k}^2\left(2 + \frac{\bar{k}}{\mathcal{G}_{\phi\phi}}\frac{d\mathcal{G}_{\phi\phi}}{dk}\right) + 2(-\bar{t})^a\right].$$

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